On the complexity of quantified integer programming

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Abstract

Quantified integer programming is the the problem of deciding assertions of the form $\mathrm{Q}_k \boldsymbol{x}_k \ldots \forall \boldsymbol{x}_2 \exists \boldsymbol{x}_1 : A \cdot \boldsymbol{x} \geq \boldsymbol{c}$ where vectors of variables $\boldsymbol{x}_k, \ldots, \boldsymbol{x}_1$ form the vector $\boldsymbol{x}$, all variables are interpreted over $\mathbb{N}$ (alternatively, over $\mathbb{Z}$), and A and $\boldsymbol{c}$ are a matrix and vector over $\mathbb{Z}$ of appropriate sizes. We show in this paper that quantified integer programming with alternation depth k is complete for the kth level of the polynomial hierarchy.

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1 Introduction

The problem of integer programming is, given a system of linear inequalities $A \cdot \boldsymbol{x} \geq \boldsymbol{b}$, to decide whether there exists a solution for $\boldsymbol{x}$ in the non-negative integers. This problem has been studied for decades, and its 0–1 version (in which the components of $\boldsymbol{x}$ are constrained to be either 0 or 1) is one of Karp's seminal 21 NP-complete problems [8]. In this paper, we study quantified integer programming (QIP), an extension of integer programming where some of the variables can be quantified universally—so that its instances have the form

$$\mathrm{Q}_k \boldsymbol{x}_k \ \ldots \ \forall \boldsymbol{x}_2 . \ \exists \boldsymbol{x}_1 : A \cdot \boldsymbol{x} \geq \boldsymbol{c} \tag{1}$$

where $\mathrm{Q}_i \in \{\exists, \forall\}$ and $\boldsymbol{x}$ consists of all first-order variables appearing in the vectors $\boldsymbol{x}_i$.

Our main contribution is settling the complexity of QIP with k quantifier blocks (as above): we prove this problem complete for the kth level of the polynomial hierarchy, similarly to the quantified version of SAT.[1] We also show that QIP with an unbounded number of quantifier blocks is PSPACE-hard and decidable in $\mathsf{STA}(*, 2^{n^{O(1)}}, n) \subseteq \mathsf{EXPSPACE}$.[2]

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1 As in the case of quantified CNF SAT, the innermost block of universal quantifiers, if present, is disregarded; e.g., the $\forall^*\exists^*\forall^*$ fragment is complete for Π_2^{P}. So we find fragments of QIP complete for $\Sigma_1^{\mathsf{P}} = \mathsf{NP}$, Π_2^{P}, Σ_3^{P}, ... , but not for $\mathsf{coNP} = \Pi_1^{\mathsf{P}}$, Σ_2^{P}, ...

2 The complexity class $\mathsf{STA}(s(n), t(n), a(n))$ was introduced by Berman [1] and contains all decision problems that can be decided by an alternating Turing machine in time $t(n)$ using space at most $s(n)$ and alternating at most $a(n)$ times on every computation branch.

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Related work and discussion. While the decidability of QIP is immediate—it can be viewed as a syntactic fragment of Presburger arithmetic, the (decidable) first-order theory of the natural numbers with addition and order, in which matrix formulas are constrained to be conjunctions of linear inequalities—its computational complexity has been unknown. It is, of course, not difficult to see that QIP (and in fact Presburger arithmetic) is PSPACE-complete if the interpretation of every first-order variable x_i is restricted to an interval $[l_i, u_i]$ that is given as part of the input: $x_i \in [l_i, u_i]$; see, e.g., [14]. But if $x_i \in \mathbb{N}$, then the best known upper bounds seem to be $\mathsf{STA}(*, 2^{2^{n^{O(1)}}}, O(n)) \subseteq$ 2-EXPSPACE, the generic upper bound for deciding Presburger arithmetic [1], and the $(k-1)$th level of the weak EXP hierarchy for the fragment with k quantifier blocks [6]. The best known lower bound has been Π_2^{P}, established recently by the authors for Π_2-instances of QIP [3, Sec. 4.2].

It may be surprising, and certainly was to the authors, that the complexity of QIP, a natural decision problem, has not yet been established. The main reason is probably that standard quantifier-elimination and automata-based techniques—which are at the core of decision procedures for Presburger arithmetic—fail to yield tight upper bounds for QIP:

- Weispfenning shows that quantifier-elimination procedures for Presburger arithmetic run in time $2^{O(|\Phi|^{(4j)^k})}$ [17, Thm. 2.1], where $|\Phi|$ denotes the size of an input formula Φ with k quantifier blocks and at most j variables in each quantifier block, and that this upper bound is essentially tight [18, Thm. 3.1]. In particular, even the NP upper bound for standard integer programming instances (Σ_1-IP) cannot be obtained by quantifier elimination.
- Automata-based decision procedures for Presburger arithmetic do not suffice either to obtain the bounds for QIP that we establish in this paper. Klaedtke shows [9, Thm. 4.6] that the size of the minimal deterministic finite automaton (DFA) for a formula Φ is upper-bounded by $2^{|\Phi|^{(j+1)^{(k+4)}}}$, which does not give any complexity bounds asymptotically better than those obtained via quantifier elimination.
- Yet another approach to QIP is to construct the semi-linear representation of the set of solutions to the system of linear inequalities of the matrix formula, and then to repeatedly project and complement this set. By an application of [2, Thm. 21], this approach gives a Π_2^{P} upper bound for the Π_2-fragment of QIP; however, as every complementation step increases the number of generators of semi-linear sets by one exponential, this approach would only yield a non-elementary upper bound for general QIP instances and fail to place fragments with bounded alternation depth inside PSPACE.

Our main results are, in short, obtained by means of a new quantifier elimination procedure on *hybrid linear sets*, which are semi-linear sets that represent sets of solutions to systems of linear inequalities. While *existential* projection ($L \mapsto \{x : \exists y.\ (x, y) \in L\}$) is a trivial operation on semi-linear sets (in generator representation), in this paper we define a dual operation, which we call *universal projection* ($L \mapsto \{x : \forall y.\ (x, y) \in L\}$), and show that its application enables us to eliminate blocks of universal quantifiers without resorting to double complementation ($\forall = \neg\exists\neg$; this would lead to a non-elementary blowup). We spell out (these and other) results of the paper in more detail in Section 3 and outline the techniques in Section 4.

Concurrently with our work and building upon a theorem of Kannan [7], Nguyen and Pak [11] have shown that Presburger arithmetic with fixed number of variables *and* fixed Boolean structure of the matrix formula (and, by necessity, where the total number of occurrences of atomic predicates is fixed) can be solved in polynomial time.

2 Preliminaries

By $\mathbb{Z}$ and $\mathbb{N}$ we denote the sets of integers and non-negative integers, respectively. Given sets X and Y, we denote by $X \Rightarrow Y$ the set of all functions with domain X and co-domain Y. Let $\mathcal{X}$ be a countably infinite set of first-order variables, and with no loss of generality assume some total ordering $\prec$ on $\mathcal{X}$. Given a finite set $X \subseteq \mathcal{X}$, an *X-indexed integer vector* is a function $\boldsymbol{v} \in (X \Rightarrow \mathbb{Z})$, and an *$X$-indexed non-negative integer vector* is a function $\boldsymbol{v} \in (X \Rightarrow \mathbb{N})$. We often call $\boldsymbol{v}$ just an *integer vector* respectively a *(non-negative) vector* when X is clear from the context. Due to the total ordering on $\mathcal{X}$, we can interchangeably write $\boldsymbol{v}$ as a tuple $(v_1, \ldots, v_n) \in \mathbb{Z}^n$ such that $n = |X|$. We denote by $\boldsymbol{e}_i$ the ith unit vector (mapping the ith variable to 1 and all other variables to 0). Addition and multiplication of a vector by a scalar value are defined component-wise. Given a set of non-negative vectors $V \subseteq \mathbb{N}^n$, its *complement* is defined as $\overline{V} := \{\boldsymbol{w} \in \mathbb{N}^n : \boldsymbol{w} \notin V\}$.

A *vector of (first-order) variables over $X \subseteq \mathcal{X}$* is a tuple $\boldsymbol{y} = (y_1, \ldots, y_\ell) \in X^\ell$ such that each $y_i \in X$ and $y_i \prec y_{i+1}$. For $\boldsymbol{v}_i \in (X_i \Rightarrow \mathbb{Z})$, $i \in \{1, 2\}$, with $X_1 \cap X_2 = \emptyset$, by $\boldsymbol{v}_1 \circ \boldsymbol{v}_2$ we denote the vector from $(X_1 \cup X_2) \Rightarrow \mathbb{Z}$ that agrees with $\boldsymbol{v}_i$ on X_i for both $i \in \{1, 2\}$. Given a vector $\boldsymbol{v} \in (X \Rightarrow \mathbb{N})$ and a vector of variables $\boldsymbol{y} = (y_1, \ldots, y_\ell) \in X^\ell$, the *projection of $\boldsymbol{v}$ removing variables $\boldsymbol{y}$* is the vector $\pi_{\boldsymbol{y}}(\boldsymbol{v}) \in ((X \setminus \{y_1, \ldots, y_\ell\}) \Rightarrow \mathbb{N})$ such that $\pi_{\boldsymbol{y}}(\boldsymbol{v})(x) := \boldsymbol{v}(x)$ for all $x \in X \setminus \{y_1, \ldots, y_\ell\}$. This definition of projection naturally extends to sets of vectors:

$$\pi_{\boldsymbol{y}}(V) := \bigcup_{\boldsymbol{v} \in V} \{\pi_{\boldsymbol{y}}(\boldsymbol{v})\} = \{\boldsymbol{v}_1 : \text{there is a } \boldsymbol{v}_2 \in (\{y_1, \ldots, y_\ell\} \Rightarrow \mathbb{N}) \text{ such that } \boldsymbol{v}_1 \circ \boldsymbol{v}_2 \in V\}.$$

For sets of vectors $V \subseteq (X \Rightarrow \mathbb{N})$, we additionally define the *universal projection*

$$\pi^*_{\boldsymbol{y}}(V) := \overline{\pi_{\boldsymbol{y}}(\overline{V})} = \{\boldsymbol{v}_1 : \text{for all } \boldsymbol{v}_2 \in (\{y_1, \ldots, y_\ell\} \Rightarrow \mathbb{N}) \text{ the vector } \boldsymbol{v}_1 \circ \boldsymbol{v}_2 \text{ is in } V\}.$$

For a vector $\boldsymbol{v} \in \mathbb{Z}^n$, we denote by $\|\boldsymbol{v}\| := \max\{\max_{x \in X} |\boldsymbol{v}(x)|, 2\}$ the *maximum norm* of $\boldsymbol{v}$. For $V \subseteq \mathbb{Z}^n$, we define $\|V\| := \max_{\boldsymbol{v} \in V} \|\boldsymbol{v}\|$. For a matrix A, we define $\|A\|$ to be the norm of its set of column vectors.

Quantified integer programming (QIP). Let A be an $n \times m$ integer matrix, $\boldsymbol{x} = (x_1, \ldots, x_m) \in X^m$ a vector of first-order variables for some finite $X \subseteq \mathcal{X}$, and $\boldsymbol{c} \in \mathbb{Z}^n$. We call $\mathfrak{S}\colon A \cdot \boldsymbol{x} \geq \boldsymbol{c}$ a *system of linear inequalities*. A *solution* to $\mathfrak{S}$ is a vector $\boldsymbol{v} \in (X \Rightarrow \mathbb{Z})$ such that $A \cdot \boldsymbol{v} \geq \boldsymbol{c}$, where "$\geq$" is interpreted component-wise. We denote by $[\![\mathfrak{S}]\!] \subseteq (X \Rightarrow \mathbb{N})$ the set of *all non-negative solutions* to $\mathfrak{S}$.

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ be vectors of first-order variables over disjoint sets of variables $X_1, \ldots, X_k$, and let $\mathfrak{S}\colon A \cdot \boldsymbol{x} \geq \boldsymbol{c}$ be a system of linear inequalities. A *formula of* QIP is given by

$$\psi = \mathrm{Q}_k \boldsymbol{x}_k.\ \mathrm{Q}_{k-1} \boldsymbol{x}_{k-1}\ \ldots\ \mathrm{Q}_1 \boldsymbol{x}_1 : A \cdot \boldsymbol{x} \geq \boldsymbol{c},$$

where $\mathfrak{S}\colon A \cdot \boldsymbol{x} \geq \boldsymbol{c}$ is a system of linear inequalities as above, $Q_i \in \{\exists, \forall\}$, and $Q_i \neq Q_{i+1}$ for all $1 \leq i < k$, i.e., quantifiers alternate between blocks of variables. The *size* $|\psi|$ of ψ is the number of bits required to write down ψ, where we assume binary encoding of numbers, and also that $|\psi| \geq \max\{2, n + m, \log\|A\|, \log\|\boldsymbol{c}\|\}$. The set $[\![\psi]\!] \subseteq (X \setminus (X_1 \cup \cdots \cup X_k) \Rightarrow \mathbb{N})$ of *non-negative solutions* to ψ is inductively defined as follows:

- for $k = 0$, $[\![\psi]\!] := [\![\mathfrak{S}]\!]$;
- for $k > 0$ and $\psi = \exists \boldsymbol{x}_k.\psi_k$, $[\![\psi]\!] := \pi_{\boldsymbol{x}_k}[\![\psi_k]\!]$; and
- for $k > 0$ and $\psi = \forall \boldsymbol{x}_k.\psi_k$, $[\![\psi]\!] := \pi^*_{\boldsymbol{x}_k}[\![\psi_k]\!]$.

A set $M \subseteq (X \Rightarrow \mathbb{N})$ is QIP-*definable* if there is a QIP-formula ψ such that $M = [\![\psi]\!]$. Whenever $X \subseteq X_1 \cup \cdots \cup X_k$, we say that ψ is a *sentence*. In this case, ψ is *valid* if $[\![\psi]\!] = \{\top\}$ where $\top$ denotes the unique function from $\emptyset$ to $\mathbb{N}$, and *invalid* if $[\![\psi]\!] = \emptyset$. If $X \setminus (X_1 \cup \cdots \cup X_k) = Y = \{y_1, \ldots, y_m\}$, we write $\psi(y_1, \ldots, y_m)$ to indicate that ψ is *open* in Y. Given $a_1, \ldots, a_m \in \mathbb{N}$, we write $\psi[a_1/x_1, \ldots, a_m/x_m]$ to denote the instance of QIP obtained from replacing every occurrence of x_i by a_i in $\mathfrak{S}$. We say that two QIP formulas ψ and ϕ are *equivalent* if $[\![\psi]\!] = [\![\phi]\!]$; note that we may always assume with no loss of generality that ψ and ϕ are open in the same set of variables.

A *(valid) instance* of the QIP problem is a (valid) sentence ψ. We call such a ψ an instance of Σ_k-IP if $Q_k = \exists$, and an instance of Π_k-IP if $Q_k = \forall$. The *alternation depth* of ψ is the number k of quantifier blocks.

Hybrid linear and semi-linear sets. Given finite sets $B, P = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\} \subseteq \mathbb{N}^m$ called *base* and *period vectors*, the *hybrid linear set generated by B and P* is the set

$$L(B, P) := \{\boldsymbol{b} + \lambda_1 \cdot \boldsymbol{p}_1 + \cdots + \lambda_n \cdot \boldsymbol{p}_n : \boldsymbol{b} \in B, \lambda_i \in \mathbb{N}, 1 \leq i \leq n\}\,.$$

The representation of $L(B, P)$ as the pair B, P (written explicitly) is called the *generator representation*. If B is singleton then $L(B, P)$ is called a *linear set*; a finite union of (hybrid) linear sets is called a *semi-linear set*. For a hybrid linear set in the generator representation $L = L(B, P)$, we denote $\|L\| := \max(\max\|B\|, \max\|P\|)$.

Hybrid linear sets represent sets of solutions to systems of linear inequalities and equalities. The following bounds on the norm in the generator representation follow from [12, Cor. 1] and [2, Prop. 4].

▶ **Proposition 1.** *Let $\mathfrak{S}\colon A \cdot \boldsymbol{x} \geq \boldsymbol{c}$ be a system of linear inequalities such that A is an $n \times m$ integer matrix. Then $[\![\mathfrak{S}]\!] = L(B, P)$ such that $\|B\|, \|P\| \leq (m \cdot \|A\| + \|\boldsymbol{c}\| + 2)^{n+m}$.*

3 Summary

The main result of this paper is the following theorem.

▶ **Theorem 2.** *Σ_k-IP is complete for Σ_k^{P} if k is odd, and Π_k-IP is complete for Π_k^{P} if k is even.*

What happens if the parity of k is different? In this case the innermost quantifiers are universal, and it turns out that they can be eliminated in a trivial way.

▶ **Corollary 3.** *Σ_{k+1}-IP is complete for Σ_k^{P} if k is odd, and Π_{k+1}-IP is complete for Π_k^{P} if k is even.*

The lower bound of Theorem 2 is proved by a reduction from an alternating version of the subset sum problem, which is essentially shown complete for the respective levels of the polynomial-time hierarchy by Travers [15]. Our reduction and more details are given in Section 7.

The upper bound of Theorem 2 is more challenging. Note that in the well-known case of Σ_1-IP, i.e., of the standard integer programming, in order to prove membership of the problem in NP, one needs to obtain polynomial upper bounds on the bit size of minimal solutions to systems of integer linear inequalities. Such bounds were derived by, e.g., von zur Gathen and Sieveking [16]. In our work, we build upon these bounds and generalize them from Σ_1-IP instances to QIP instances.

▶ **Proposition 4** (small witness property). *For a* QIP *instance* ψ *of the form* (1) *with* k *quantifier blocks, the validity of* ψ *does not change if variables of the vector* $\boldsymbol{x}_k$ *(bound by the quantifiers of the outermost block) are interpreted over* $[0, M-1]$ *instead of* $\mathbb{N}$*, where* $\log M = |\psi|^{O(k)}$.

The domains of other variables can then be bounded in turn as follows—which places QIP with fixed alternation depth into PH.

▶ **Proposition 5** (relativization-type theorem). *For a* QIP *instance* ψ *of the form* (1) *with* k *quantifier blocks, the validity of* ψ *does not change if, for each* $i \in [1,k]$*, all variables of the vector* $\boldsymbol{x}_i$ *(bound by the quantifiers of the* i*th innermost block) are interpreted over* $[0, M_i - 1]$ *instead of* $\mathbb{N}$*, where* $\log M_i = |\psi|^{O(2k-i)}$ *and the constant of* $O(\cdot)$ *is independent of* ψ*,* k*, and* i.

Let us point out that in Proposition 5 it is not possible to substitute $[0, M-1]$ for the range of *all* variables; not only using $M = \max M_i$, but in fact using any finite M. For example, the sentence $\forall x.\, \exists y : y = x + 1$ is true if x and y are interpreted in $\mathbb{N}$, but false if they are interpreted in any finite segment $[0, M-1]$.

▶ Remark 6. The last observation, of course, also holds for Presburger arithmetic in general: any relativization-type theorem (analogous to Proposition 5) must assign different ranges to variables from different quantifier blocks; for instance, this reveals a flaw in the formulation of the relativization-type Theorem 2.2 in [17].

Notice that our small witness property (Proposition 4) is specific to QIP, in the sense that its bound is smaller by one exponential compared to its analogue for general Presburger formulas [17, Thm. 2.2] (the latter is, in fact, tight, as shown implicitly in, e.g., [5, 6]). At the core of our small witness property is a new quantifier elimination procedure for QIP:

▶ **Proposition 7** (quantifier elimination). *Given a* QIP *formula* $\phi(\boldsymbol{x})$ *with alternation depth* k*, there exists an equivalent* Σ_1-IP *formula* $\phi'(\boldsymbol{x})$ *with at most* $2^{|\psi|^{O(k)}}$ *existentially quantified variables and numbers of absolute value bounded by* $2^{|\psi|^{O(k)}}$.

The ideas behind Propositions 4 and 7 are outlined in the following Section 4.

Further results. Our results give a uniform upper bound for the general QIP problem, where the number of quantifier blocks can be unbounded. For such a QIP instance, our relativization-type theorem (Proposition 5) suggests doubly exponential ranges for all variables, which places QIP in the complexity class $\mathsf{STA}(*, 2^{n^{O(1)}}, n)$, as $k \le n$. The best lower bound is PSPACE, by the arguments of Section 7.

Another by-product of our techniques is a pseudo-polynomial algorithm for QIP in which the total number of variables is fixed and the matrix formula is $A \cdot \boldsymbol{x} = \boldsymbol{c}$ instead of $A \cdot \boldsymbol{x} \ge \boldsymbol{c}$.

In terms of auxiliary techniques, on the way to our quantifier elimination procedure for QIP we discover (in Sections 5 and 6) some new properties of hybrid linear sets. In particular, these properties enable us to find, as a side result, a polynomial-time algorithm for universality of hybrid linear sets in the generator representation, even if all input numbers are written in binary (Proposition 18 in Section 5).

Finally, our results extend in a natural way to the version of quantified integer programming where all variables are interpreted over $\mathbb{Z}$ instead of over $\mathbb{N}$: the results of Theorem 2 and Corollary 3 still hold.

4 Main ideas

As explained in Section 3, bounding the range of the outermost quantifier is the main technical task in our development. In this section we explain how to do this, thus sketching the ideas behind both the small witness property (Proposition 4) and the quantifier elimination procedure (Proposition 7).

Suppose we start with a QIP instance ψ of the form (1); to find a suitable upper bound M_k for the range of the $\boldsymbol{x}_k$ variables of ψ, we will compute generator representations for the sets of models of formulas

$$\psi_j(\boldsymbol{x}_k, \ldots, \boldsymbol{x}_{j+1}) = \mathrm{Q}_j \boldsymbol{x}_j \ \ldots \ \forall \boldsymbol{x}_2.\ \exists \boldsymbol{x}_1 : A \cdot \boldsymbol{x} \geq \boldsymbol{c}$$

for all $j \in [0, k]$, where, as previously, $\boldsymbol{x}$ is the concatenation of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$. For each value of the parameter j, we will find upper bounds on the integers appearing in these representations, starting with $j = 0$ and culminating with $j = k$. The upper bound for the value of parameter $j = k$ will be a valid choice for M_k.

Let us now describe this computation in more detail. Consider a simple abstract example, a Σ_3-IP instance with 3 variables, $\psi\colon \exists x.\ \forall y.\ \exists z : A \cdot \boldsymbol{x} \geq \boldsymbol{c}$ where $\boldsymbol{x} = (x, y, z)$. Let $L_0 \subseteq \mathbb{N}^3$ be the set of all models of $\psi_0\colon A \cdot \boldsymbol{x} \geq \boldsymbol{c}$; this is a hybrid linear set—denote it $L(C_0, Q_0)$—with $\|C_0\|$, $\|Q_0\|$ upper-bounded by a polynomial in $\|A\|, \|\boldsymbol{c}\|$ with degree at most the size of ψ (see, e.g., Proposition 1). It follows that $\log\|C_0\|$ and $\log\|Q_0\|$ are polynomial in the size of ψ. It is clear that the the set $L_1 = [\![\psi_1]\!] = \{(x, y) \in \mathbb{N}^2 : \text{there exists a } z \in \mathbb{N} \text{ such that } (x, y, z) \in L_0\}$ is simply a projection of $L(C_0, Q_0)$, and in particular $L_1 = L(C, Q)$ where the sets C and Q are obtained by removing z-coordinates from all vectors in C_0 and Q_0, respectively. Hence, $\log\|C\|$ and $\log\|Q\|$ are also polynomial in the size of ψ. (This will, of course, work for all occurrences of the existential quantifier in ψ, including $\exists x$ in our example; but we will need to handle the universal quantifier $\forall y$ before handling $\exists x$.)

The next step is to transform the generator representation $L(C, Q)$ of the set $L_1 = [\![\psi_1]\!]$ into a generator representation of the set

$$L_2 = [\![\psi_2]\!] = \{x \in \mathbb{N} : \text{for all } y \in \mathbb{N} \text{ it holds that } (x, y) \in L_1\}.$$

This set L_2 is the *universal projection* of $L(C, Q)$: $L_2 = \pi_y^*(L(C, Q))$; cf. Section 2. As the main technical contribution of the present paper, we show that, in general, (*i*) universal projections of hybrid linear sets are hybrid linear sets themselves and that (*ii*) universal projection as an operation on hybrid linear sets can only lead to a moderate increase in the magnitude of generators. (These results are summarized in Proposition 11 below. For the usual projection, such facts are obvious.)

We now briefly introduce the techniques that we develop for handling the universal projection. Define for each $y \in \mathbb{N}$ the cross section $S(y) = \{x \in \mathbb{N} : (x, y) \in L_1\}$, then

$$L_2 = \bigcap_{y \in \mathbb{N}} S(y) \tag{2}$$

by definition. Each set $S(y)$ is a semi-linear set (and, in fact, a hybrid linear set—because it is essentially the intersection of two hybrid linear sets, see Lemma 15, and such intersections are hybrid linear sets themselves, see, e.g., [2, Theorem 6]), but the intersection in (2) is infinite, and, in general, an infinite intersection of semi-linear sets does not have to be semi-linear.[3] However, we prove (in Section 5) the following lemma, which is our first and key insight:

[3] For every $n \geq 1$, consider the hybrid linear set $L_n = \mathbb{N} \setminus \{0, n\} = L([1, n-1] \cup \{2n\}, \{n\})$. Given any $A \subseteq \mathbb{N}$, the intersection $\bigcap_{n \in A} L_n = \mathbb{N} \setminus (\{0\} \cup A)$ is only semi-linear (i.e., ultimately periodic) if so is A.

▶ **Lemma 8.** *Let $L = L(C, Q) \subseteq \mathbb{N}^m$ be a hybrid linear set with $C, Q \subseteq \mathbb{N}^m$. Let the components of vectors be indexed by m variables X, let $U \subseteq X$, $|U| = s$, and suppose $\boldsymbol{u}$ is the corresponding vector of variables. Then the following statements hold:*

- *If, for some variable $u_i \in U$, the set Q contains no multiple of the unit vector $\boldsymbol{e}_i$ associated with u_i, then $\pi^*_{\boldsymbol{u}}(L) = \emptyset$.*
- *Otherwise, denote $a_i = \min\{a : a \cdot \boldsymbol{e}_i \in Q\}$ and $H = \{\boldsymbol{b} \in \mathbb{N}^s : 0 \le \boldsymbol{b}(u_i) \le a_i - 1$ for all $u_i \in U\}$. Then*

$$\pi^*_{\boldsymbol{u}}(L) = \bigcap_{\boldsymbol{b} \in H} \pi_{\boldsymbol{u}}\big(L(C, Q) \cap \{\boldsymbol{u} = \boldsymbol{b}\}\big)$$

where $\{\boldsymbol{u} = \boldsymbol{b}\}$ denotes the hybrid linear set $\{\boldsymbol{c} \in \mathbb{N}^m : \boldsymbol{c}(u_i) = \boldsymbol{b}(u_i)$ for all $u_i \in U\}$.

In other words, unless $L_2 = \emptyset$, the intersection in (2) can be made finite without changing its result: $\bigcap_{y \in \mathbb{N}} S(y) = \bigcap_{y < N} S(y)$, where $\log N$ is polynomial in the size of ψ. Since, as we have just mentioned, hybrid linear sets are closed under finite intersections, this shows that the set L_2 is hybrid linear, and, in fact, the following general result follows:

▶ **Proposition 9.** *A set in $\mathbb{N}^m$ is* QIP*-definable iff it is hybrid linear.*

Furthermore, the set L_2 turns out to have a small generator representation as well. Indeed, we first observe that all sets $S(y)$ have representations $L(B_y, P)$ with a common set of periods P and with $\|B_y\|$, $\|P\|$ small if so is $\|y\|$ (Lemma 15 in Section 5). We then prove (in Section 6) the following lemma, which is our second insight:

▶ **Lemma 10.** *Let $L_i = L(C_i, Q)$, $i \in [1, n]$, be hybrid linear sets with $C_i, Q \subseteq \mathbb{N}^m$. The set $S = \bigcap_{i=1}^n L_i$ has a representation $S = L(B, Q)$ where $\|B\| \le \max_{i \in [1,n]} \|L_i\|^{O(m^3)}$ independently of n.*

In other words, long intersections of hybrid linear sets with a common set of periods preserve small representations, regardless of the number of sets in the intersection. Combining Lemmas 8 and 10, we obtain the following statement:

▶ **Proposition 11.** *Let $L = L(C, Q) \subseteq \mathbb{N}^m$ be a hybrid linear set with $C, Q \subseteq \mathbb{N}^m$. Let the components of vectors be indexed by m variables $\boldsymbol{u}, \boldsymbol{v}$, and suppose the vector $\boldsymbol{u}$ has s variables. Then the universal projection $\pi^*_{\boldsymbol{u}}(L)$ has a representation $L(B, P)$ where $P = \pi_{\boldsymbol{u}}(\{\boldsymbol{q} \in Q : q_1 = \ldots = q_s = 0\})$ and $\|B\| \le \|L\|^{O(m^5)}$.*

In particular, we conclude that the set $L_2 = \bigcap_{y < N} L(B_y, P)$ has a representation $L(B, P)$ with $\|B\| < M$ where $\log M$ is polynomial in the size of ψ. But note that $\psi = \psi_3$ is true iff $L_2 = [\![\psi_2]\!]$ is non-empty; therefore, the validity of ψ is unchanged if the range of $\exists x$ is changed from $\mathbb{N}$ to $[0, M - 1]$. Thus, in our example the bound M_3 can be chosen as M; it can hence be deduced that a Σ_3^{P} algorithm can handle such instances. The argument for the general case follows the same lines.

5 Universal projection and universality

A semi-linear set in $\mathbb{N}^d$ is called *universal* if it is equal to $\mathbb{N}^d$.

▶ **Example 12.** A one-dimensional hybrid linear set $L = L(B, P) \subseteq \mathbb{N}$ with $B, P \subseteq \mathbb{N}$ is universal iff $P \setminus \{0\} \neq \emptyset$ and L contains the integer segment $[0, k - 1]$ where $k = \min P \setminus \{0\}$. Indeed, the right-to-left direction is immediate: if L satisfies the conditions above, then $\mathbb{N} = L([0, k - 1], \{k\}) \subseteq L$. For the left-to-right direction, suppose $L = \mathbb{N}$. First observe that the set $P \setminus \{0\}$ is non-empty because L is infinite. Therefore, $k > 0$ is well-defined. Second, as $L = \mathbb{N}$, the set L contains all natural numbers, in particular those in $[0, k - 1]$.

The following lemma generalizes Example 12; recall that $\boldsymbol{e}_i$ denotes the ith unit vector.

▶ **Lemma 13.** *A hybrid linear set $L = L(B, P) \subseteq \mathbb{N}^m$ with $B, P \subseteq \mathbb{N}^m$ is universal iff P contains vectors $a_i \cdot \boldsymbol{e}_i$ for some $a_i > 0$, for every $i \in [1, m]$, and L contains the box $H = [0, a_1 - 1] \times \ldots \times [0, a_m - 1]$.*

Proof. The right-to-left direction is immediate: if L satisfies the conditions of the lemma, then $\mathbb{N}^m = L(H, \{a_1 \cdot \boldsymbol{e}_1, \ldots, a_m \cdot \boldsymbol{e}_m\}) \subseteq L$. For the left-to-right direction, suppose $L = \mathbb{N}^m$. We first prove that, for each $i \in [1, m]$, the set of periods P contains a vector $a_i \cdot \boldsymbol{e}_i$ with $a_i > 0$. Assume without loss of generality that $i = 1$ and denote $N = \mathbb{N} \times \boldsymbol{0} \subseteq \mathbb{N}^m$. Since L is universal (and $\mathbb{Q}_{\geq 0} \times \boldsymbol{0}$ is a face of $\mathbb{Q}^m_{\geq 0}$), $N = L(B, P) \cap N \subseteq L(B \cap N, P \cap N)$. Therefore, the set $P \cap N$ contains at least one vector $a_1 \cdot \boldsymbol{e}_1$ with $a_1 > 0$, otherwise the set N would be finite. Hence, P contains $a_1 \cdot \boldsymbol{e}_1, \ldots, a_m \cdot \boldsymbol{e}_m$ with all $a_i > 0$. It now remains to note that, as $L = \mathbb{N}^m$, the set L contains all nonnegative integer vectors, in particular those in H. This completes the proof. ◀

▶ Remark 14. If $m = 1$, then in the statement of Lemma 13, the condition $H \subseteq L(B, P)$ is equivalent to the condition $H \subseteq B$, as long as H is defined using the *shortest* vector $a_1 \cdot \boldsymbol{e}_1$ in $P \setminus \{\boldsymbol{0}\}$. For $m \geq 2$, this is no longer the case.

▶ **Lemma 15.** *Suppose $L = L(C, Q) \subseteq \mathbb{N}^m$ and $M = L(D, E) \subseteq \mathbb{N}^m$ where $E = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_s\}$. Then the set $L \cap M$ has a representation $L(B, P)$ where $P = \{\boldsymbol{q} \in Q : q_{s+1} = \ldots = q_m = 0\}$ and $\|B\| \leq \|L\|^{O(m^2)} \cdot \|M\|^{O(m)}$.*

We can now restate and prove Lemma 8, which appeared previously in Section 4.

▶ **Lemma 8.** *Let $L = L(C, Q) \subseteq \mathbb{N}^m$ be a hybrid linear set with $C, Q \subseteq \mathbb{N}^m$. Let the components of vectors be indexed by m variables X, let $U \subseteq X$, $|U| = s$, and suppose $\boldsymbol{u}$ is the corresponding vector of variables. Then the following statements hold:*

- *If, for some variable $u_i \in U$, the set Q contains no multiple of the unit vector $\boldsymbol{e}_i$ associated with u_i, then $\pi^*_{\boldsymbol{u}}(L) = \emptyset$.*
- *Otherwise, denote $a_i = \min\{a : a \cdot \boldsymbol{e}_i \in Q\}$ and $H = \{\boldsymbol{b} \in \mathbb{N}^s : 0 \leq \boldsymbol{b}(u_i) \leq a_i - 1$ for all $u_i \in U\}$. Then*

$$\pi^*_{\boldsymbol{u}}(L) = \bigcap_{\boldsymbol{b} \in H} \pi_{\boldsymbol{u}}\big(L(C, Q) \cap \{\boldsymbol{u} = \boldsymbol{b}\}\big)$$

where $\{\boldsymbol{u} = \boldsymbol{b}\}$ denotes the hybrid linear set $\{\boldsymbol{c} \in \mathbb{N}^m : \boldsymbol{c}(u_i) = \boldsymbol{b}(u_i)$ for all $u_i \in U\}$.

Proof. Denote $V = X \setminus U$; we will abuse notation and let symbols $\boldsymbol{u}$ and $\boldsymbol{v}$ refer to U- and V-indexed integer vectors (wherever this creates no confusion). By definition, a vector $\boldsymbol{v}^*$ belongs to $\pi^*_{\boldsymbol{u}}(L)$ if and only if for all $\boldsymbol{u}$ the vector $(\boldsymbol{u}, \boldsymbol{v}^*)$ belongs to L. This condition is equivalent to the requirement that

$$L \cap \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{N}^m : \boldsymbol{v} = \boldsymbol{v}^*\} = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{N}^m : \boldsymbol{v} = \boldsymbol{v}^*\}. \tag{3}$$

Note that $\{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{N}^m : \boldsymbol{v} = \boldsymbol{v}^*\} = L((\boldsymbol{0}, \boldsymbol{v}^*), E)$ where E is the set of all unit vectors associated with variables $\boldsymbol{u}$. We now apply Lemma 15: $L \cap L((\boldsymbol{0}, \boldsymbol{v}^*), E) = L(D_{\boldsymbol{v}^*}, R)$ where $R = \{\boldsymbol{q} = (\boldsymbol{u}, \boldsymbol{v}) \in Q : \boldsymbol{v} = \boldsymbol{0}\}$. Now the requirement (3) has the form $L(D_{\boldsymbol{v}^*}, R) = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{N}^m : \boldsymbol{v} = \boldsymbol{v}^*\}$ and, by Lemma 13, is equivalent to the requirement that, first, the set R contains some multiple of the unit vector, $a_i \cdot \boldsymbol{e}_i$ for some $a_i > 0$, associated with each variable $u_i \in U$, and, second, the set $L(D_{\boldsymbol{v}^*}, R)$ contains the box

$$H(\boldsymbol{v}^*) = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{N}^m : \boldsymbol{v} = \boldsymbol{v}^*, 0 \leq u_i \leq a_i - 1 \text{ for all variables } u_i \in U\}.$$

Note that in the statement of Lemma 13 we can always choose $a_i \cdot \boldsymbol{e}_i$ to be the shortest vectors of the required form in R; expanding the definition of R then gives

$$a_i = \min\{a : a \cdot \boldsymbol{e}_i \in Q\} \qquad \text{for each variable } u_i \in U. \tag{4}$$

We now make the following observations. First, the set R does not depend on the vector $\boldsymbol{v}^*$, but only on Q and on the way the variables are split into $\boldsymbol{u}$ and $\boldsymbol{v}$. Therefore, the condition that R contains $a_i \cdot \boldsymbol{e}_i$ for some $a_i > 0$ is either satisfied or not satisfied for all $\boldsymbol{v}^*$ simultaneously. In the former case, $\pi^*_{\boldsymbol{u}}(L) = \emptyset$; so it suffices to consider the latter case. We have the following equivalence:

$$\boldsymbol{v}^* \in \pi^*_{\boldsymbol{u}}(L) \quad \text{iff} \quad (\boldsymbol{u}, \boldsymbol{v}^*) \in L(D_{\boldsymbol{v}^*}, R) \text{ for all } \boldsymbol{u} \in H$$

where $H = \{\boldsymbol{u} : 0 \le u_i \le a_i - 1 \text{ for all variables } u_i \in U\}$ and a_i are as defined in (4). Since $L(D_{\boldsymbol{v}^*}, R)$ was chosen as $L \cap \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{N}^m : \boldsymbol{v} = \boldsymbol{v}^*\}$, this is the same as

$$\boldsymbol{v}^* \in \pi^*_{\boldsymbol{u}}(L) \quad \text{iff} \quad (\boldsymbol{u}, \boldsymbol{v}^*) \in L \text{ for all } \boldsymbol{u} \in H,$$

and the equation of the lemma follows. ◀

▶ **Example 16.** Consider any set $L = L(C, \{3\,\boldsymbol{e}_2\}) \subseteq \mathbb{N}^2$ with a finite $C \subseteq \mathbb{N}^2$. Its universal projection $L' = \pi^*_y(L) = \{x : (x, y) \in L \text{ for all } y \in \mathbb{N}\}$ can be obtained by taking cross sections $S_b = \{x : (x, b) \in L\}$ for $b = 0, 1, 2$, removing the y coordinate, and intersecting the results: $L' = \pi_y(S_0) \cap \pi_y(S_1) \cap \pi_y(S_2)$ where the projection $\pi_y \colon \mathbb{N}^2 \to \mathbb{N}$ removes the y coordinate. So whether or not a specific $a \in \mathbb{N}$ belongs to L' is fully determined by whether the vectors $(a, 0)$, $(a, 1)$, and $(a, 2)$ belong to L. In fact, this conclusion will also hold if instead of L we consider any set $M = L(C, \{3\,\boldsymbol{e}_2\} \cup Q)$ where Q contains no vectors of the form $a \cdot \boldsymbol{e}_2$.

Intermezzo: Deciding universality of hybrid linear sets

The technique developed above, in fact, enables us to show that universality of hybrid linear sets (given in generator representation) can be decided in polynomial time, even if all numbers are written in binary. Consider the following lemma, which is a more general version of Example 12 and Lemma 13.

▶ **Lemma 17.** *Let $L = L(B, P) \subseteq \mathbb{N}^m$ be a hybrid linear set with $B, P \subseteq \mathbb{N}^m$. Define the set of* shallow points*,*

$$W = \{\, \boldsymbol{w} \in \mathbb{N}^m : \textit{there is no } \boldsymbol{p} \in P \setminus \{\boldsymbol{0}\} \textit{ with } \boldsymbol{w} \ge \boldsymbol{p} \,\} = \mathbb{N}^m \setminus \bigcup_{\boldsymbol{p} \in P \setminus \{\boldsymbol{0}\}} (\boldsymbol{p} + \mathbb{N}^m).$$

Then L is universal iff $W \subseteq B$.

Indeed, for Example 12, observe that for $m = 1$ the set W is the integer segment $[0, k-1]$ where $k = \min P \setminus \{0\}$; cf. Remark 14.

For Lemma 13, note that $W \subseteq B$ is only possible if W is finite, which implies that for each $i \in [1, m]$ there is a vector $a_i \cdot \boldsymbol{e}_i \in P$ with $a_i > 0$ (otherwise all such vectors for some given i are in W, and there are infinitely many of them). But then $W \subseteq H = [0, a_1 - 1] \times \ldots \times [0, a_m - 1]$.

▶ **Proposition 18.** *There is a polynomial-time algorithm that takes a hybrid linear set $L(B, P) \subseteq \mathbb{N}^m$, presented as $B, P \subseteq \mathbb{N}^m$ with numbers written in binary, and decides if $L(B, P)$ is universal.*

Proof. By the characterization of Lemma 17, it is sufficient to check if $W \subseteq B$. First check that the necessary condition of Lemma 13 is satisfied: if for some i there is no $a_i \cdot \boldsymbol{e}_i \in P$ with $a_i > 0$, then $L(B, P)$ is not universal. Otherwise consider the Hasse diagram of the partial order $(H, \leq)$, i.e., the directed acyclic graph with vertex set H and all edges $(\boldsymbol{x}, \boldsymbol{y})$ where $\boldsymbol{x} < \boldsymbol{y}$ and there is no $\boldsymbol{z}$ with $\boldsymbol{x} < \boldsymbol{z} < \boldsymbol{y}$. Notice that this graph does not have to be of polynomial size with respect to the input.

Run the depth-first search (DFS) procedure on this graph, starting from $\boldsymbol{0}$ and for each $\boldsymbol{x} \in H$ ordering the outgoing edges $(\boldsymbol{x}, \boldsymbol{y})$ according to the (unique) index $i \in [1, m]$ for which $x_i < y_i$. Whenever the current node is outside W, the algorithm backtracks (observe that the set W is always downward closed, i.e., whenever $\boldsymbol{w} \in W$ and $\boldsymbol{w}' \leq \boldsymbol{w}$, then also $\boldsymbol{w}' \in W$); if it is in W but not in B, the algorithm terminates immediately, reporting that $L(B, P)$ is not universal. If the search finishes, the algorithm concludes that $W \subseteq B$ and reports that $L(B, P)$ is universal. All visited nodes are marked and not re-entered, ensuring that no node is ever visited twice. As all visited notes are checked for inclusion in B, which is given as part of the input, it follows that the running time of the search is proportional to the size of the input, and the entire procedure works in polynomial time. ◀

6 Long intersections

▶ **Lemma 19.** *Let $L_i = L(C_i, Q)$, $i \in [1, n]$, be hybrid linear sets with $C_i, Q \subseteq \mathbb{N}^m$. Suppose the vectors of Q are linearly independent. Then the set $S = \bigcap_{i=1}^{n} L_i$ has a representation $S = L(B, Q)$ where $\|B\| \leq 2^{O(m \log m)} \cdot \max_{i \in [1,n]} \|L_i\| \cdot \|Q\|^m$ independently of n.*

Proof (sketch). Note that $\bigcap_{i=1}^{n} L(C_i, Q)$ is the union over all $\boldsymbol{c}_1 \in C_1, \ldots, \boldsymbol{c}_n \in C_n$ of $\bigcap_{i=1}^{n} L(\boldsymbol{c}_i, Q)$, so we shall assume with no loss of generality that $C_i = \{\boldsymbol{c}_i\}$ for all i.

Define a point lattice $\mathcal{L} = Q \cdot \mathbb{Z}^r = \{Q \cdot \boldsymbol{u} : \boldsymbol{u} \in \mathbb{Z}^r\}$ where $r = |Q|$; see, e.g., [10, Chapter 2]. Vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^m$ are called *congruent* modulo $\mathcal{L}$, written $\boldsymbol{x} \equiv \boldsymbol{y} \pmod{\mathcal{L}}$, if and only if $\boldsymbol{x} - \boldsymbol{y} \in \mathcal{L}$. This congruence splits $\mathbb{Z}^m$ into a disjoint union of equivalence classes, which have the form $\boldsymbol{x} + \mathcal{L}$ where $\boldsymbol{x} \in \mathbb{Z}^m$. Note that the relation $\equiv$ is compatible with addition and subtraction of elements of Q, in the sense that vectors $\boldsymbol{x} \pm \boldsymbol{q}$, $q \in Q$, belong to the same equivalence class as $\boldsymbol{x}$; therefore, $L_i = \boldsymbol{c}_i + Q \cdot \mathbb{N}^r \equiv \boldsymbol{c}_i \pmod{\mathcal{L}}$. Hence, unless the intersection $\bigcap_{i=1}^{n} L_i$ is empty, it must be the case that $\boldsymbol{c}_i \equiv \boldsymbol{c}_j \pmod{\mathcal{L}}$ for all i, j. We assume in the sequel that this is indeed the case, i.e., all sets L_i are contained in the same equivalence class $\boldsymbol{c}_1 + \mathcal{L}$.

Let us now define the coordinates in $\boldsymbol{c}_1 + \mathcal{L}$ in a natural way. Consider the mapping $\psi \colon \boldsymbol{c}_1 + \mathcal{L} \to \mathbb{Z}^r$ that maps each $\boldsymbol{x}$ into a vector $\boldsymbol{u} = \psi(\boldsymbol{x})$ such that $\boldsymbol{x} = \boldsymbol{c}_1 + Q \cdot \boldsymbol{u}$; note that $\boldsymbol{u}$ exists as long as $\boldsymbol{x} \in \boldsymbol{c}_1 + \mathcal{L}$ and is determined uniquely because the vectors in Q are linearly independent. The mapping ψ is, in fact, a bijection between $\boldsymbol{c}_1 + \mathcal{L}$ and $\mathbb{Z}^r$, so $L_1 \cap \ldots \cap L_n = \psi^{-1}(\psi(L_1) \cap \ldots \psi(L_n))$. Denote $\boldsymbol{f}_i = \psi(\boldsymbol{c}_i)$ and observe that $\psi(L_i) = \psi(\boldsymbol{c}_i) + \mathbb{N}^r$. So a vector $\boldsymbol{v} \in \mathbb{Z}^r$ belongs to the intersection of all $\psi(L_i)$ if and only if $\boldsymbol{v} \geq \boldsymbol{f}_i$ for all $i \in [1, n]$. This condition is satisfied if and only if $\boldsymbol{v} \geq \boldsymbol{f}$ where $\boldsymbol{f}$ is the component-wise maximum of vectors $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n$; in other words, $\bigcap_{i=1}^{n} \psi(L_i) = \boldsymbol{f} + \mathbb{N}^r$ and $L_1 \cap \ldots \cap L_n = L(\psi^{-1}(\boldsymbol{f}), Q)$.

It remains to find an upper bound on $\|\psi^{-1}(\boldsymbol{f})\|$. Note that $\psi^{-1}(\boldsymbol{f}) = \boldsymbol{c}_1 + Q \cdot \boldsymbol{f}$, so $\|\psi^{-1}(\boldsymbol{f})\| \leq \|\boldsymbol{c}_1\| + \|Q \cdot \boldsymbol{f}\|$. Suppose $\boldsymbol{f} = (f^1, \ldots, f^r)$ and $Q = \{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_r\}$, then $Q \cdot \boldsymbol{f} = f^1 \cdot \boldsymbol{q}_1 + \ldots + f^r \cdot \boldsymbol{q}_r$. Recall that each f^j is, in fact, a component of some $\boldsymbol{f}_i = \psi(\boldsymbol{c}_i)$.

For this $i = i(j)$ it holds that $\boldsymbol{c}_i = \boldsymbol{c}_1 + Q \cdot \boldsymbol{f}_i$, and by [2, Proposition 3] we have

$$|f^j| \leq 2^{O(m \log m)} \cdot \max\left(\|\boldsymbol{c}_i - \boldsymbol{c}_1\|, \|Q\|\right) \cdot \|Q\|^{m-1} \quad \text{and}$$
$$\|\psi^{-1}(\boldsymbol{f})\| \leq \|C_1\| + m \cdot \max_{j \in [1,r]} \|f^j \cdot \boldsymbol{q}_j\| \leq 2^{O(m \log m)} \cdot \max_{i \in [1,n]} \|L_i\| \cdot \|Q\|^m. \quad \blacktriangleleft$$

We can now restate and prove Lemma 10, which appeared previously in Section 4.

▶ **Lemma 10.** *Let $L_i = L(C_i, Q)$, $i \in [1,n]$, be hybrid linear sets with $C_i, Q \subseteq \mathbb{N}^m$. The set $S = \bigcap_{i=1}^n L_i$ has a representation $S = L(B, Q)$ where $\|B\| \leq \max_{i \in [1,n]} \|L_i\|^{O(m^3)}$ independently of n.*

Proof (sketch). We first apply a discrete version of the Carathéodory theorem [2, Proposition 5] to the set L_1, decomposing it into a union of hybrid linear sets with linearly independent periods:

$$L_1 = \bigcup_j M_j \quad \text{where} \quad M_j = L(D_j, Q_j) \quad \text{and} \quad \|D_j\| \leq \|C_1\| + (\#Q \cdot \|Q\|)^{O(m)},$$

with each $Q_j \subseteq Q$ a set of linear independent vectors (here and below $\#$ denotes the cardinality of a set). The intent is to make it possible to invoke Lemma 19.

Notice that, whereas intersecting two hybrid linear sets L and L' with sets of periods P and $P' \subseteq P$, respectively, will always give a hybrid linear set with the set of periods P' (see, e.g., [2, Theorem 6] and, transitively, Theorem 5.6.1 of [4, p. 180]), this observation would not suffice for our purposes. Indeed, the magnitude of the base vectors in the hybrid linear representation of $L \cap L'$ can still increase compared to the magnitude of the base vectors of L and L'; and so $n-1$ consecutive applications of this operation would lead to a blowup in the representation size if n grows. Instead of using this observation, we will rely on Lemma 19 to defeat the effect of large n, and will use another trick to make its application possible.

Indeed, observe that

$$L_1 \cap L_2 \cap \ldots \cap L_n = \bigcup_j M_j \cap L_2 \cap \ldots \cap L_n = \bigcup_j (M_j \cap L_2) \cap \ldots \cap (M_j \cap L_n).$$

Since the sets of periods of M_j and L_i are Q_j and Q, respectively, it follows by [2, Theorem 6] that each $M_j \cap L_i$ is a hybrid linear set with representation $L(B_{i,j}, Q_j)$, where

$$\|B_{i,j}\| \leq ((\#Q_j + \#Q) \cdot \max(\|M_j\|, \|L_i\|))^{O(m)} \leq \max\left(\|C_1\| + (\#Q \cdot \|Q\|)^{O(m)}, \|L_i\|\right)^{O(m)}.$$

But now, for each j, the intersection of $L(B_{i,j}, Q_j)$, $i \in [2,n]$, satisfies the conditions of Lemma 19, and thus can be written as $L(B_j, Q_j)$ with $\|B_j\|$ small with respect to $\|B_{i,j}\|$ and $\|Q_j\|$ (estimations to follow). Now $S = \bigcup_j L(B_j, Q_j)$, and it remains to note that, as $L_i + L(\mathbf{0}, Q) = L_i$ for all i, it also holds that $S + L(\mathbf{0}, Q) = S$ and hence

$$S = \bigcup_j L(B_j, Q_j) + L(\mathbf{0}, Q) = \bigcup_j L(B_j, Q) = L\left(\bigcup_j B_j,\ Q\right), \quad \text{with}$$
$$\|B_j\| \leq 2^{O(m \log m)} \cdot \max\left(\max_{i \in [2,n]} \|B_{i,j}\|, \|Q_j\|\right) \cdot \|Q_j\|^m \leq \max_{i \in [1,n]} \|L_i\|^{O(m^3)}. \quad \blacktriangleleft$$

7 Lower bounds

We show lower bounds for QIP and Σ_k-IP via a reduction from a generalisation of the classical SubsetSum problem. For odd k, let $\boldsymbol{a}_k \in \mathbb{N}^{m_k}, \ldots, \boldsymbol{a}_1 \in \mathbb{N}^{m_1}$ be vectors of natural

numbers, and let $t \in \mathbb{N}$ be a target. An instance of Σ_k-SubsetSum is a tuple $(\boldsymbol{a}_k, \ldots, \boldsymbol{a}_1, t)$. This instance is a valid instance whenever the following holds:

$$\exists \boldsymbol{x}_k \in \{0,1\}^{m_k}. \forall \boldsymbol{x}_{k-1} \in \{0,1\}^{m_{k-1}} \ldots \exists \boldsymbol{x}_1 \in \{0,1\}^{m_1} : \sum_{i=1}^{k} \boldsymbol{a}_i \cdot \boldsymbol{x}_i = t. \tag{5}$$

Thus, Σ_k-SubsetSum can be viewed as the 0–1 variant of Σ_k-IP, i.e., variables are only interpreted over $\{0,1\}$. For even k, Π_k-SubsetSum is defined analogously. When we take the union of Σ_k-SubsetSum for all $k > 0$, we obtain QSubsetSum.

▶ **Proposition 20.** *For every fixed $k > 0$, for odd k Σ_k-SubsetSum is Σ_k^{P}-complete, and for even k Π_k-SubsetSum is Π_k^{P}-complete.* QSubsetSum *is* PSPACE*-complete.*

Upper bounds for Σ_k-SubsetSum and QSubsetSum can be obtained trivially. The PSPACE lower bound for QSubsetSum was established by Travers in [15, Lem. 4]. Unfortunately, the construction given in [15] does not directly yield Σ_k^{P} hardness for Σ_k-SubsetSum, as the lower bound for QSubsetSum is shown in [15] by a reduction from 3-CNF QBF in which the alternating quantifiers range over *single* variables, and Σ_k^{P} hardness for 3-CNF k-QBF requires an unbounded number of variables in every quantifier block [13]. It is not difficult to show that the construction from [15] can indeed be adapted in order to yield Σ_k^{P} hardness for Σ_k-SubsetSum for odd k, and likewise for even k.

Proof of lower bounds in Theorem 2

We reduce from Σ_k-SubsetSum and show how to transform an instance given as (5) into an equivalent instance of Σ_k-IP. Note that the existentially quantified variables do not present an issue, since, for instance, $\boldsymbol{x}_1 \in \{0,1\}^{m_1}$ iff $\boldsymbol{x}_1 \leq \mathbf{1}$, i.e., (5) is equivalent to

$$\exists \boldsymbol{x}_k \in \{0,1\}^{m_k}. \forall \boldsymbol{x}_{k-1} \in \{0,1\}^{m_{k-1}} \ldots \forall \boldsymbol{x}_2 \in \{0,1\}^{m_2}. \exists \boldsymbol{x}_1 : \sum_{i=1}^{k} \boldsymbol{a}_i \cdot \boldsymbol{x}_i = t \wedge \boldsymbol{x}_1 \leq \mathbf{1}. \tag{6}$$

The key insight is that, for universally quantified variables, conjunctions of linear integer constraints can express division with remainder using any fixed divisor. In particular, consider

$$\exists \boldsymbol{x}_k \in \{0,1\}^{m_k}. \forall \boldsymbol{x}_{k-1} \in \{0,1\}^{m_{k-1}} \ldots \forall \boldsymbol{x}_2. \exists \boldsymbol{x}_1. \exists \boldsymbol{\lambda} : \\ \sum_{i=3}^{k} \boldsymbol{a}_i \cdot \boldsymbol{x}_i + \boldsymbol{a}_2 \cdot (\boldsymbol{x}_2 - 2 \cdot \boldsymbol{\lambda}) + \boldsymbol{a}_1 \cdot \boldsymbol{x}_1 = t \wedge \boldsymbol{x}_1 \leq \mathbf{1} \wedge \mathbf{0} \leq \boldsymbol{x}_2 - 2 \cdot \boldsymbol{\lambda} \leq \mathbf{1}. \tag{7}$$

We claim that the sentences (6) and (7) are equivalent. First, no matter what $\boldsymbol{x}_2$ is, $\boldsymbol{\lambda}$ has to be $\lfloor \boldsymbol{x}_2/2 \rfloor$ in order to satisfy the last constraint of (7). If sentence (6) is true, then (7) is also true. Indeed, if $\boldsymbol{x}_2 \in \{0,1\}^{m_2}$, then we can choose $\boldsymbol{\lambda} = \mathbf{0}$ and the inequalities become the same as before (and thus, for instance, there is an appropriate $\boldsymbol{x}_1$). Analogously, if $\boldsymbol{x}_2$ is outside $\{0,1\}^m$, then it is the vector $\boldsymbol{x}_2 - 2 \cdot \boldsymbol{\lambda}$ that is in $\{0,1\}^{m_2}$, and for this vector we already know the appropriate $\boldsymbol{x}_1$ from the previous formula. Conversely, suppose the sentence (7) is true, then it is in particular true for all choices $\boldsymbol{x}_2 \in \{0,1\}^{m_2}$ in which case $\boldsymbol{\lambda} = \lfloor \boldsymbol{x}_2/2 \rfloor = \mathbf{0}$. Hence, the assignment for $\boldsymbol{x}_1$ chosen in (7) given $\boldsymbol{x}_2$ will also work for (6). This proves the claim.

In fact, the trick above works regardless of how many universal variables we have and at which positions they occur in the quantifier prefix. So we can handle both existential and universal variables and can transform any instance of Σ_k-SubsetSum respectively Π_k-SubsetSum into an equivalent instance of Σ_k-IP respectively Π_k-IP, which yields the desired lower bounds, when variables are interpreted over the natural numbers.

References

1 Leonard Berman. The complexitiy of logical theories. *Theor. Comput. Sci.*, 11:71–77, 1980. doi:10.1016/0304-3975(80)90037-7.

2 Dmitry Chistikov and Christoph Haase. The taming of the semi-linear set. In *Automata, Languages, and Programming, ICALP*, volume 55 of *LIPIcs*, pages 128:1–128:13. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016. doi:10.4230/LIPIcs.ICALP.2016.128.

3 Dmitry Chistikov, Christoph Haase, and Simon Halfon. Context-free commutative grammars with integer counters and resets. *Theor. Comput. Sci.*, pages –, 2017. To appear. doi:10.1016/j.tcs.2016.06.017.

4 Seymour Ginsburg. *The mathematical theory of context-free languages*. McGraw-Hill, 1966.

5 Erich Grädel. Dominoes and the complexity of subclasses of logical theories. *Ann. Pure Appl. Logic*, 43(1):1–30, 1989. doi:10.1016/0168-0072(89)90023-7.

6 Christoph Haase. Subclasses of Presburger arithmetic and the weak EXP hierarchy. In *Computer Science Logic and Logic in Computer Science, CSL-LICS*, pages 47:1–47:10. ACM, 2014. doi:10.1145/2603088.2603092.

7 Ravi Kannan. Test sets for integer programs, ∀∃ sentences. In *Polyhedral Combinatorics, Proceedings of a DIMACS Workshop, Morristown, New Jersey, USA, June 12-16, 1989*, pages 39–48, 1990.

8 Richard M. Karp. Reducibility among combinatorial problems. In *Complexity of Computer Computations*, The IBM Research Symposia Series, pages 85–103. Plenum Press, New York, 1972.

9 Felix Klaedtke. Bounds on the automata size for Presburger arithmetic. *ACM Trans. Comput. Log.*, 9(2):11:1–11:34, 2008. doi:10.1145/1342991.1342995.

10 Jiri Matoušek. *Lectures on discrete geometry*. Graduate texts in mathematics. Springer, 2002. doi:10.1007/978-1-4613-0039-7.

11 Danny Nguyen and Igor Pak. Complexity of short Presburger arithmetic. In *49th Annual ACM Symposium on the Theory of Computing, STOC*, 2017. To appear.

12 Loïc Pottier. Minimal solutions of linear Diophantine systems: Bounds and algorithms. In *Rewriting Techniques and Applications, RTA*, volume 488 of *Lect. Notes Comp. Sci.*, pages 162–173. Springer, 1991. doi:10.1007/3-540-53904-2_94.

13 Larry J. Stockmeyer. The polynomial-time hierarchy. *Theor. Comput. Sci.*, 3(1):1–22, 1976. doi:10.1016/0304-3975(76)90061-X.

14 K. Subramani. Tractable fragments of Presburger arithmetic. *Theory Comput. Syst.*, 38(5):647–668, 2005. doi:10.1007/s00224-004-1220-0.

15 Stephen D. Travers. The complexity of membership problems for circuits over sets of integers. *Theor. Comput. Sci.*, 369(1-3):211–229, 2006. doi:10.1016/j.tcs.2006.08.017.

16 Joachim von zur Gathen and Malte Sieveking. A bound on solutions of linear integer equalities and inequalities. *P. Am. Math. Soc.*, 72(1):155–158, 1978. doi:10.1090/S0002-9939-1978-0500555-0.

17 Volker Weispfenning. The complexity of almost linear Diophantine problems. *J. Symb. Comp.*, 10(5):395–403, 1990. doi:10.1016/S0747-7171(08)80051-X.

18 Volker Weispfenning. Complexity and uniformity of elimination in Presburger arithmetic. In *Symbolic and Algebraic Computation, ISSAC*, pages 48–53. ACM, 1997. doi:10.1145/258726.258746.